

An operator summability of sequences in Banach spaces

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Abstract

Let $1 \leq p < \infty$. A sequence $\langle x_n \rangle$ in a Banach space X is defined to be p -operator summable if for each $\langle f_n \rangle \in l_p^{w*}(X^*)$, we have $\langle \langle f_n(x_k) \rangle_k \rangle_n \in l_p^s(l_p)$. Every norm p -summable sequence in a Banach space is operator p -summable, while in its turn every operator p -summable sequence is weakly p -summable. An operator $T \in B(X, Y)$ is said to be p -limited if for every $\langle x_n \rangle \in l_p^w(X)$, $\langle Tx_n \rangle$ is operator p -summable. The set of all p -limited operators form a normed operator ideal. It is shown that every weakly p -summable sequence in X is operator p -summable if and only if every operator $T \in B(X, l_p)$ is p -absolutely summing. On the other hand every operator p -summable sequence in X is norm p -summable if and only if every p -limited operator in $B(l_{p'}, X)$ is absolutely p -summing. Moreover, this is the case if and only if X is a subspace of $L_p(\mu)$ for some Borel measure μ .

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1 Introduction

Let X be a Banach space, $\langle x_n \rangle$ a sequence in X and $1 \leq p < \infty$. We say that $\langle x_n \rangle$ is (norm) p -summable in X if $\sum_{n=1}^{\infty} \|x_n\|^p < \infty$. If $\sum_{n=1}^{\infty} |f(x_n)|^p < \infty$, for all $f \in X^*$, then we say that $\langle x_n \rangle$ is weakly p -summable in X . It is easy to note that a norm p -summable sequence is always a weakly p -summable, while the converse, in general, is not true. In fact in a Banach space X every weakly p -summable sequence is norm p -summable if and only if X is finite dimensional. These two types of summability were used by Grothendieck [10] to introduce the operator ideal of absolutely summing operators (for $p = 1$), further generalized by Pietsch [14] who defined the operator ideal of absolutely p -summing operators for all $1 \leq p < \infty$. These operator ideals have been studied extensively in the literature.

Let $l_p^s(X)$ denote the set of all norm p -summable sequences and $l_p^w(X)$ that of all weakly p -summable sequences in X . Then these two sets become Banach spaces under suitable norms. More precisely, $l_p^s(X)$ can be identified as the 'countable p -direct sum' of X ; similarly, $l_p^w(X)$ can be shown to be isometrically isomorphic to the space $B(l_{p'}, X)$ of operators if $p > 1$ (here p' is the harmonic conjugate of p , i.e. $\frac{1}{p} + \frac{1}{p'} = 1$) and $l_{p'}$ is replaced by c_0 when $p = 1$.

In this paper we introduce a new kind of summability of sequences in Banach spaces using the notion of p -summing operators and call it the operator p -summability (definition below). This notion crops up naturally while extending the idea of limited sets to a p -level. In general, this type of summability of sequences is different from both weak and norm summability. In this paper, we investigate Banach spaces for which this type of summability coincides either with weak or with norm summability. For the first type of Banach spaces in question, we encounter a p -level of Dunford-Pettis property whereas for the other we are encouraged to introduce the notion of a p -level of Gelfand-Phillips property. The later type of Banach space ultimately reduces to subspaces of $L_p(\mu)$ for some Borel measure μ .

Example of a Banach space can be constructed for which the operator p -summability is different from both norm as well as weak p -summabilities.

2 An operator summability

A non-empty subset S of a Banach space X is said to be limited if for every weak*-null sequence $\langle f_n \rangle$ in X^* (i.e., $\lim_{n \rightarrow \infty} f_n(x) = 0$, for all $x \in X$), $f_n \rightarrow 0$ uniformly on S . Alternatively, given a weak*-null sequence $\langle f_n \rangle$ in X^* there is an $\langle \alpha_n \rangle \in c_0$ such that $|f_n(x)| \leq \alpha_n$ for all $x \in S$ and all $n \in \mathbf{N}$. We can extend this idea to the ' p -sense' in the following way. We define a subset S of X to be p limited in X ($1 \leq p < \infty$) if for every weak*- p -summable sequence $\langle f_n \rangle$ in X^* (i.e., $\sum_{n=1}^{\infty} |f_n(x)|^p < \infty$ for all $x \in X$) there is an $\langle \alpha_n \rangle \in l_p$ such that $|f_n(x)| \leq \alpha_n$ for all $x \in S$ and $n \in \mathbf{N}$.

The history of limited sets originated from the following error of Gelfand [9]: A set S in Banach space X is compact if and only if every weak*-null sequence in X^* is uniformly null on S . Clearly, every compact set has this property. However, Phillips [15] came out with an example of a non-compact set with the above property, i.e., of a limited non-compact set. The authors [18] recently, studied the concept of p -compact sets for $1 \leq p < \infty$. It is interesting to note that the above mentioned analogy carries over to p -level too. First we show that p -compact sets are p -limited.

We begin with some definitions. For $x = \langle x_n \rangle \in l_p^w(X)$, we define an operator $E_x : l_{p'} \rightarrow X$ given by $E_x(\alpha) = \sum_{n=1}^{\infty} \alpha_n x_n$, $\alpha = \langle \alpha_n \rangle \in l_{p'}$. Then $E_x \in B(l_{p'}, X)$. Moreover, in this identification $l_p^w(X)$ is isometrically isomorphic to $B(l_{p'}, X)$. For $p = 1$, $l_{p'}$ is replaced by c_0 . We say that $K \subset X$ is relatively p -compact if there is an $x = \langle x_n \rangle \in l_p^s(X)$ such that $K \subset E_x(Ball(l_{p'}))$. Similarly, $K \subset X$ is said to be (relatively) weakly p -compact if there is an $x = \langle x_n \rangle \in l_p^w(X)$ such that $K \subset E_x(Ball(l_{p'}))$.

Proposition 2.1 *Let $1 \leq p < \infty$ and X a Banach space. Then every p -compact subset of X is p -limited.*

Proof. Let $K \subset X$ be a p -compact and $\langle f_n \rangle \in l_p^{w*}(X^*)$. There is an $x = \langle x_k \rangle \in l_p^s(X)$ such that $K \subset E_x(Ball(l_{p'}))$. Then

$$\sum_{k=1}^{\infty} \sum_{n=1}^{\infty} |f_n(x_k)|^p \leq (\|\langle f_n \rangle\|_p^{w*})^p \sum_{k=1}^{\infty} \|x_k\|^p,$$

so that

$$\sum_{n=1}^{\infty} \sum_{k=1}^{\infty} |f_n(x_k)|^p = \sum_{k=1}^{\infty} \sum_{n=1}^{\infty} |f_n(x_k)|^p \leq (\|\langle f_n \rangle\|_p^{w*} \|x\|_p^s)^p.$$

Set $(\sum_{k=1}^{\infty} |f_n(x_k)|^p)^{\frac{1}{p}} = \alpha_n$ for all n so that $\langle \alpha_n \rangle \in l_p$. Now if $z \in K$, then $z = \sum_{k=1}^{\infty} \beta_k x_k$ for some $\langle \beta_k \rangle \in \text{Ball}(l_{p'})$, and for each n we have

$$|f_n(z)| = \left| \sum_{k=1}^{\infty} \beta_k f_n(x_k) \right| \leq \left(\sum_{k=1}^{\infty} |\beta_k|^{p'} \right)^{\frac{1}{p'}} \left(\sum_{k=1}^{\infty} |f_n(x_k)|^p \right)^{\frac{1}{p}} \leq \alpha_n.$$

Hence, K is p -limited. \triangle

Next we observe certain facts about p -limited sets.

Proposition 2.2 *Let A and B be two subsets of a Banach space X .*

- (a) *If B is p -limited and $A \subset B$, then A is also p -limited.*
- (b) *If A is p -limited then \overline{A} is p -limited.*
- (c) *If if A and B are p -limited sets then so are $A \cup B$, $A + B$ and $A \cap B$.*
- (d) *If A is p -limited and $T \in B(X, Y)$, then $T(A)$ is p -limited in Y .*

Proof. Suppose A is p -limited. We prove (b). Let $\langle f_n \rangle \in l_p^{w*}(X^*)$. Then there is an $\langle \alpha_n \rangle \in l_p$ such that $|f_n(x)| \leq \alpha_n$ for all $x \in A$ and $n \in \mathbb{N}$. Let $x \in \overline{A}$. Then there is an $\langle x_k \rangle$ in A such that $x_k \rightarrow x$. Thus for each n , $f_n(x_k) \rightarrow f_n(x)$. Fixing n we have $|f_n(x_k)| \leq \alpha_n$ for all k . It follows that $|f_n(x)| \leq \alpha_n$ for all n , so that \overline{A} is p -limited. Thus (b) follows. The proofs of (a), (c) and (d) are immediate. \triangle

Lemma 2.3 *Let $\langle x_n \rangle \in l_p^w(X)$. Then $E_x(\text{Ball}(l_{p'}))$ is p -limited if and only if for every $\langle f_n \rangle \in l_p^{w*}(X^*)$, we have $\langle \langle f_n(x_k) \rangle_k \rangle_n \in l_p^s(l_p)$.*

Proof. Consider $x = \langle x_n \rangle \in l_p^w(X)$ such that $S = E_x(\text{Ball}(l_{p'}))$ is p -limited. Then given $\langle f_n \rangle \in l_p^{w*}(X^*)$, there is an $\langle \alpha_n \rangle \in l_p$ such that for each $\beta = \langle \beta_k \rangle \in \text{Ball}(l_{p'})$ we have

$$|f_n(E_x(\beta))| \leq \alpha_n, \text{ for all } n.$$

i.e.

$$|\langle \beta, \langle f_n(x_k) \rangle \rangle_{k=1}^{\infty}| \leq \alpha_n, \text{ for all } n.$$

It follows that $\|\langle f_n(x_k) \rangle_{k=1}^{\infty}\|_p \leq \alpha_n$, for all n . Thus $\langle \langle f_n(x_k) \rangle_k \rangle_n \in l_p^s(l_p)$ for all $\langle f_n \rangle \in l_p^{w*}(X^*)$.

Now tracing back the proof, we can prove the converse also. \triangle

Since $l_p^{w*}(X^*)$ can be identified with $B(X, l_p)$, where to each $f \in \langle f_n \rangle \in l_p^{w*}(X^*)$ we get $(E_f)_* \in B(X, l_p)$ given by $(E_f)_*(x) = \langle f_n(x) \rangle$ with $\|f\|_p^{w*} = \|(E_f)_*\|$, and since $l_p^{w*}(X^*) = l_p^w(X^*)$ the above lemma can be reorganized as follows.

Proposition 2.4 *Let $x = \langle x_n \rangle$ be a sequence in X . The following are equivalent.*

- (a) $x \in l_p^w(X)$ and $E_x(\text{Ball}(l_{p'}))$ is a p -limited set in X .
- (b) $\langle Tx_n \rangle \in l_p^s(l_p)$ for all $T \in B(X, l_p)$.
- (c) $E_x \in \Pi_p^d(l_{p'}, X)$.

In this way we observe a new notion of summability in Banach spaces in the ‘ p -sense’. Let us rename this phenomena as follows:

Definition 2.5 *A sequence $\langle x_n \rangle$ in X is said to be operator p -summable in X if it satisfies one (and hence all) of the conditions of Proposition 2.4.*

Note that every norm p -summable sequence in X is operator p -summable. To see this, let $\langle x_n \rangle \in l_p^s(X)$ and $T \in B(X, l_p)$. Then $\|Tx_n\| \leq \|T\|\|x_n\|$ for all n so that $\langle Tx_n \rangle \in l_p^s(l_p)$. Thus $\langle x_n \rangle$ is operator p -summable. We have already seen that an operator p -summable sequence is weakly p -summable.

3 Towards weak summability

In this section we characterize Banach spaces with the property that every weakly p -summable sequence is operator p -summable and give some examples of such spaces. We shall call a Banach space with this property a *weak p -space*. A simple characterization of such spaces in terms of operator ideals is given below. Let X and Y be Banach spaces and let $1 \leq p < \infty$. Then an operator $T \in B(X, Y)$ is called absolutely p -summing if for every $\langle x_n \rangle \in l_p^w(X)$, $\langle Tx_n \rangle \in l_p^s(Y)$. The set of all absolutely p -summing operators in $B(X, Y)$ is denoted by $\Pi_p(X, Y)$.

Proposition 3.1 *Let X be a Banach space and $1 \leq p < \infty$. Then X is a weak p -space if and only if $\Pi_p(X, l_p) = B(X, l_p)$.*

Proof. Let $T \in B(X, l_p)$ and $x = \langle x_n \rangle \in l_p^w(X)$. Suppose X is a weak p -space. Then $\langle x_n \rangle$ is operator p -summable so that $\langle Tx_n \rangle \in l_p^s(l_p)$. Thus $T \in \Pi_p(X, l_p)$. Tracing back we can prove the converse. \triangle

Before we give some examples of weak p -spaces we shall further explore Banach spaces that satisfy an operator ideal equation of the above type. Given Banach spaces X and Y , let $W(X, Y)$ and $\nu(X, Y)$ denote the sets of weakly compact and completely continuous operators from X to Y respectively. Recall that a Banach space X is said to have the Dunford-Pettis property (D. P. P., for short) if for any Banach space Y , $W(X, Y) \subset \nu(X, Y)$.

In 1940, Dunford and Pettis [8] proved that every weakly compact operator defined on a $L_1(\mu)$ space takes weakly compact sets to norm compact sets. In 1953, Grothendieck [11] defined a Banach space X to have the Dunford-Pettis property if weakly compact operators defined on X are completely continuous and proved that $C(K)$ spaces also have this property. This result was also obtained independently in 1955 by Bartle, Dunford and Schwartz [1]. Brace [2] and Grothendieck [11] gave some nice characterizations of the Dunford-Pettis property. A detailed survey of Dunford-Pettis property can be found in [6]. In this section we propose to extend this property to a p -setting to meet our above mentioned end. For this purpose we recall the following characterization of D.P.P. essentially due to Grothendieck [11].

Theorem 3.2 *Let X be a Banach space then the following statements are equivalent.*

- (a) $W(X, Y) \subset \nu(X, Y)$ for all Banach spaces Y .
- (b) $W(X, c_0) \subset \nu(X, c_0)$.
- (c) For $\langle x_n \rangle \in c_0^w(X)$ and $\langle f_n \rangle \in c_0^w(X^*)$, $\langle \langle f_k(x_n) \rangle_k \rangle_n \in c_0^s(c_0)$.
- (d) For $\langle x_n \rangle \in c_0^w(X)$ and $\langle f_n \rangle \in c_0^w(X^*)$, $\langle f_n(x_n) \rangle \in c_0$.

Picking up (c) as an end, we now propose the following definition.

Definition 3.3 *Let $1 \leq p, q \leq \infty$. A Banach space X is said to have the (p, q) -Dunford-Pettis property ((p, q) -D.P.P., for short) if given $\langle x_n \rangle \in l_q^w(X)$ and $\langle f_n \rangle \in l_p^w(X^*)$, we have $\langle \langle f_k(x_n) \rangle_k \rangle_n \in l_q^s(l_p)$. For p (or q) = ∞ , l_p (or l_q) is replaced by c_0 . For all p , the (p, p) -Dunford-Pettis property shall be called the p -Dunford-Pettis property.*

It is immediate from Theorem 3.2 that the ∞ -D.P.P. is the classical Dunford-Pettis property. In what follows we shall extend the above characterization theorem to the (p, q) -setting. Towards this end the notion of weak p -compactness studied by the authors [18] (also see Castillo [4, 5]) fits snugly in the scheme. Let X and Y be Banach spaces and let $1 \leq p < \infty$. An operator $T \in B(X, Y)$ is said to be p -compact (weakly p -compact) if $T(Ball(X))$ is relatively p -compact (respectively, relatively weakly p -compact). Here l_p is replaced by c_0 if $p = 1$. Let $W_p(X, Y)$ denote the set of all weakly p -compact operators and $K_p(X, Y)$ that of all p -compact operators. The next result was obtained by the authors [18].

Theorem 3.4 *Let X and Y be Banach spaces, $1 \leq p < \infty$ and $T \in B(X, Y)$. Then the following statements are equivalent:*

- (a) *T is weakly p -compact.*
- (b) *There are $y \in l_p^w(Y)$ and $S_y \in B(R(y), X^*)$ such that $T^* = S_y \cdot E_y^*$, where $R(y) = \overline{Range(E_y^*)} \subset l_p$.*

The set $W_p(X, Y)$ of all weakly p -compact operators in $B(X, Y)$ is a Banach operator ideal with the factorization norm ω_p defined as follows:

$$\omega_p(T) = \inf \{ \|S_y\| \|E_y\| : T^* = S_y \cdot E_y^* \text{ as in theorem 3.4(b)} \}.$$

Let (A, α) be an operator ideal. For Banach spaces X and Y we put

$$A^d(X, Y) = \{T \in B(X, Y) : T^* \in A(Y^*, X^*)\}.$$

For an operator $T \in A^d(X, Y)$, we put $\alpha^d(T) = \alpha(T^*)$. With these notations (A^d, α^d) is also a Banach operator ideal and is called the dual ideal of (A, α) .

Corollary 3.5 *For Banach spaces X and Y , $T \in W_p^d(X, Y)$ if and only if there are $f = \langle f_n \rangle \in l_p^w(X^*)$ and $S_f \in B(R(f), Y)$ such that $T = S_f \cdot (E_f)_*$. Here, $R(f) = \{\langle f_n(x) \rangle : x \in X\} \subset l_p$ and $(E_f)_* = E_f^*|_X$.*

We can now extend the classical characterization theorem for the Dunford-Pettis property to the (p, q) -setting. Let X and Y be a pair of Banach spaces and $T \in B(X, Y^*)$. Then we have $T = i_Y^* \cdot T^{**} \cdot i_X$. Indeed, for $x \in X$ and $y \in Y$ we have $\langle i_Y^* \cdot T^{**} \cdot i_X(x), y \rangle = \langle Tx, y \rangle$. Here $i_X : X \hookrightarrow X^{**}$ is the canonical embedding.

Theorem 3.6 *Let $1 \leq p < \infty$, $1 \leq q \leq \infty$ and X a Banach space. Then the following statements are equivalent:*

- (a) X has the (p, q) -Dunford-Pettis property.
- (b) $W_p(Y, X^*) \subset \Pi_q^d(Y, X^*)$ for every Banach space Y .
- (c) $\Pi_q^d(l_{p'}, X^*) = B(l_{p'}, X^*)$.

Proof. It only remains to show that (a) implies (b), for $W_p(l_{p'}, X^*) = B(l_{p'}, X^*)$. To this end, assume that X has the (p, q) -DPP and let $T \in W_p(Y, X^*)$. Then by Theorem 3.4, there are $\mathbf{f} = \langle f_n \rangle \in l_p^w(X^*)$ and $S_{\mathbf{f}} \in B(R(\mathbf{f}), Y^*)$ such that $T^* = S_{\mathbf{f}} \cdot E_{\mathbf{f}}^*$. First we show that $T^*|_X \in \Pi_q(X, Y^*)$. To see this, let $\langle x_n \rangle \in l_q^w(X)$. Then as $\langle f_n \rangle \in l_p^w(X^*)$, we have $\langle (E_{\mathbf{f}})^* \cdot i_X(x_n) \rangle = \langle \langle f_k(x_n) \rangle_k \rangle_n \in l_q^s(l_p)$. Thus $\langle T^* \cdot i_X(x_n) \rangle \in l_q^s(Y^*)$. In other words, $T^* \cdot i_X \in \Pi_q(X, Y^*)$. It follows from Proposition 2.19 in [7] that $i_X^* \cdot T^{**} \in \Pi_q^d(Y^{**}, X^*)$, so that $T = i_X^* \cdot T^{**} \cdot i_Y \in \Pi_q^d(Y, X^*)$. This completes the proof. \triangle

Note In [18] the authors have observed that absolutely p -summing operators may be regarded as p -completely continuous operators as they take weakly p -compact sets to p -compact sets. Thus the classical Dunford-Pettis property may be traced back provided we regard absolutely p -summing operators as p -completely continuous operators.

In view of Proposition 3.1 and Theorem 3.6, we have the following characterization for weak p -spaces.

Theorem 3.7 *Let $1 \leq p < \infty$. Then for a Banach space X the following statements are equivalent.*

- (a) X is a weak p -space
- (b) X has the p -Dunford-Pettis property.
- (c) $W_p(Y, X^*) \subset \Pi_p^d(Y, X^*)$ for every Banach space Y .
- (d) $\Pi_p^d(l_{p'}, X^*) = B(l_{p'}, X^*)$.
- (c') $W_p^d(X, Y^*) \subset \Pi_p(X, Y^*)$ for every Banach space Y .
- (d') $\Pi_p(X, l_p) = B(X, l_p)$

Proof. Note that $W_p^d(X, l_p) = B(X, l_p)$. Thus in the light of Proposition 3.1 and Theorem 3.6, it is enough to show that (c) \Leftrightarrow (c') and that (d') \Rightarrow (d). First assume that $W_p(Y, X^*) \subset \Pi_p^d(Y, X^*)$ and let $T \in W_p^d(X, Y^*)$. Then

$T^* \cdot i_Y \in W_p(Y, X^*) \subset \Pi_p^d(Y, X^*)$. Thus $i_Y^* \cdot T^{**} \in \Pi_p(X^{**}, Y^*)$ so that $T = i_Y^* \cdot T^{**} \cdot i_X \in \Pi_p(X, Y^*)$ [7, 2.4]. Therefore, $W_p^d(X, Y^*) \subset \Pi_p(X, Y^*)$. Next, let $W_p^d(X, Y^*) \subset \Pi_p(X, Y^*)$. If $T \in W_p(Y, X^*)$, then by Theorem 3.4, $T^* = S_f \cdot E_f^*$ for some $f \in l_p^w(X^*)$. Thus by Corollary 3.5, $T^* \cdot i_X \in W_p^d(X, Y^*) \subset \Pi_p(X, Y^*)$. It follows from Proposition 2.19 in [7], that $T = i_X^* \cdot T^{**} \cdot i_Y \in \Pi_p^d(Y, X^*)$. Thus $W_p(Y, X^*) \subset \Pi_p^d(Y, X^*)$ so that (c) \Leftrightarrow (c'). Now, let $1 < p < \infty$ and assume that $\Pi_p(X, l_p) = B(X, l_p)$. Let $T \in B(l_{p'}, X)$. Then $T^* \cdot i_X \in B(X, l_p) = \Pi_p(X, l_p)$. Thus $i_X^* \cdot T^{**} \in \Pi_p^d(l_{p'}, X^*)$. As $l_{p'}$ is reflexive, we have $T = i_X^* \cdot T^{**}$ so that $\Pi_p^d(l_{p'}, X^*) = B(l_{p'}, X^*)$. Finally, suppose that $\Pi_1(X, l_1) = B(X, l_1)$. Let $T \in B(c_0, X^*)$. Then $T^* \cdot i_X \in B(X, l_1) = \Pi_1(X, l_1)$. Thus $i_X^* \cdot T^{**} \cdot i_{c_0} \in \Pi_1^d(c_0, X^*)$ so that $\Pi_1^d(c_0, X^*) = B(c_0, X^*)$. Therefore, (d') \Rightarrow (d), which completes the proof. \triangle

Some more consequences of Theorem 3.6 are in order.

Corollary 3.8 *If $1 \leq q \leq p < \infty$ and if X has the (p, q) -Dunford-Pettis property, then it has the p -Dunford-Pettis property. In particular, X is a weak p -space.*

Corollary 3.9 *If X^* has the p -Dunford-Pettis property, then so does X . In other words if X^* is a weak p -space then so is X .*

Remark: The $p = \infty$ case of the above corollary; i.e., if X^* has the classical Dunford-Pettis property then so does X , was proved by Grothendieck [11].

Note It is interesting to note that Diestel et al. [7, p. 433] defined a Banach space X to be a Hilbert-Schmidt space if every Hilbert space operator that factors through X is a Hilbert-Schmidt operator. They observed that a Banach space X is a Hilbert-Schmidt space if and only if, for each $\langle x_n \rangle \in l_2^w(X)$ and $\langle f_n \rangle \in l_2^w(X^*)$, $\langle f_n(x_n) \rangle \in l_2$. Clearly, all Banach spaces satisfying the 2-Dunford-Pettis property are Hilbert-Schmidt spaces. However, the class of all Hilbert-Schmidt operators is clearly much larger. They proposed that, the class of Hilbert-Schmidt spaces could be studied as the class of Banach spaces satisfying the Dunford-Pettis property of “level 2”. However, we clearly see that Banach spaces satisfying the 2-Dunford-Pettis property are more appropriate than the class of Hilbert-Schmidt spaces as the former exactly mimics the classical case geometrically as well as analytically.

Examples 1. Let X be an \mathcal{L}_∞ -space. If $1 \leq p \leq 2$, then X has the $(p, 2)$ -DPP and 2 is sharp [10, 13]. If $2 < p < q < \infty$, then X has the (p, q) -DPP

and q is sharp, that is to say that X has the almost p -DPP for every $p > 2$ [16, 12].

2. In view of Theorem 3.6 above, every \mathcal{L}_1 -space has the above properties. In particular, c_0 and l_1 have the 2-DPP, the almost p -DPP if $p > 2$ and also the ∞ -DPP (= Dunford-Pettis property).

It is interesting to note that these are the only \mathcal{L}_p -spaces with any (r, s) -DPP, $1 \leq r, s \leq \infty$.

Theorem 3.10 *Let $1 < p < \infty$. Then l_p does not have the r -Dunford-Pettis property for any $r > 1$. In other words, for $1 < p < \infty$, l_p is not a weak r -space for any $r > 1$.*

Proof. We divide the proof in several parts.

Case 1. Let $r \geq \max\{p, p'\}$. Let $\{e_n\}$ be the standard unit vector basis of l_p and $\{f_n\}$ that of $l_{p'}$. Then $\langle e_n \rangle \in l_r^w(l_p)$ and $\langle f_n \rangle \in l_r^w(l_{p'})$. Since $\langle \langle f_k(e_n) \rangle_k \rangle_n = \langle \langle \delta_k^n \rangle_k \rangle_n \notin l_r^s(l_r)$, where δ_k^n is the Kronecker delta, we conclude that l_p does not have the r -DPP if $r \geq \max\{p, p'\}$.

Before we proceed to the other cases, we need to prove the following lemma.

Lemma 3.11 *Let $1 \leq s \leq p'$, where p' is the harmonic conjugate of p , $1 < p < \infty$. Find $t > s$ such that $\frac{1}{s} - \frac{1}{p'} = \frac{1}{t}$. Then for any $\langle \alpha_n \rangle \in l_t$, $\langle \alpha_n e_n \rangle \in l_s^w(l_p)$.*

Proof of the Lemma: If $\langle \beta_n \rangle \in l_{p'}$, then

$$\begin{aligned} \left(\sum_{n=1}^{\infty} |\langle \beta, \alpha_n e_n \rangle|^s \right)^{1/s} &= \left(\sum_{n=1}^{\infty} |\alpha_n \beta_n|^s \right)^{1/s} \\ &\leq \left(\sum_{n=1}^{\infty} |\alpha_n|^t \right)^{1/t} \left(\sum_{n=1}^{\infty} |\beta_n|^{p'} \right)^{1/p'} \\ &< \infty. \end{aligned}$$

Thus $\langle \alpha_n e_n \rangle \in l_s^w(l_p)$. \triangle

Now we consider the other cases of the theorem.

Case 2. Let $1 < r < \min\{p, p'\}$. Find $t_1, t_2 > 1$ such that $\frac{1}{t_1} = \frac{1}{r} - \frac{1}{p'}$ and $\frac{1}{t_2} = \frac{1}{r} - \frac{1}{p}$. Then $\frac{1}{t_1} + \frac{1}{t_2} = \frac{2}{r} - 1 < \frac{1}{r}$. Thus we can find $\langle \alpha_n \rangle \in l_{t_1}$ and

$\langle \beta_n \rangle \in l_{t_2}$ such that $\langle \alpha_n \beta_n \rangle \notin l_r$. Now by the above lemma $\langle \alpha_n e_n \rangle \in l_r^w(l_p)$ and $\langle \beta_n f_n \rangle \in l_r^w(l_{p'})$. But

$$\langle \langle \langle \beta_k f_k, \alpha_n e_n \rangle \rangle_k \rangle_n \notin l_r^s(l_r).$$

Thus l_p does not have the r -DPP if $1 \leq r < \min\{p, p'\}$.

Case 3. Let r lie between p and p' . Note that l_p has the r -DPP if and only if $l_{p'}$ has the r -DPP. Thus without any loss of generality we may assume that $p < r < p'$. Find $t > 1$ such that $\frac{1}{t} = \frac{1}{r} - \frac{1}{p'}$. Then $r < t$ so that we can find $\langle \alpha_n \rangle \in l_t$ with $\langle \alpha_n \rangle \notin l_r$. Then $\langle \alpha_n e_n \rangle \in l_r^w(l_p)$. Also $\langle f_n \rangle \in l_r^w(l_{p'})$. But

$$\langle \langle \langle f_k, \alpha_n e_n \rangle \rangle_k \rangle_n \notin l_r^s(l_r).$$

Thus l_p does not have the r -DPP if r lies between p and p' .

Finally, since $\Pi_p(l_p) \neq B(l_p)$, we conclude that both l_p and $l_{p'}$ do not have the p - and p' -DPP. This completes the proof. \triangle

Corollary 3.12 *For $1 < p < \infty$, l_p does not have*

- (a) *The (r, s) -Dunford-Pettis property if $1 < r, s < \infty$.*
- (b) *The $(r, 1)$ -Dunford-Pettis property if $1 < r < \infty$.*
- (c) *The $(1, r)$ -Dunford-Pettis property if $1 < r < \infty$.*

Proof. (a) For $1 < s \leq r$, if l_p has the (r, s) -DPP then it also has the r -DPP. Thus if $1 < s \leq r < \infty$, then l_p does not have the (r, s) -DPP.

Next, let $1 < r < s$. Find $\langle \alpha_n \rangle \in l_s^w(l_p)$ such that $\langle \alpha_n \rangle \notin l_r^w(l_p)$. Find $\beta \in l_{p'}$ such that $\sum_{n=1}^{\infty} |\langle \beta, \alpha_n \rangle|^r = \infty$. Putting $\beta_1 = \beta$ and $\beta_n = 0$ for $n \geq 2$, $\langle \beta_n \rangle \in l_r^w(l_{p'})$. However,

$$\langle \langle \langle \beta_k, \alpha_n \rangle \rangle_k \rangle_n \notin l_s^s(l_r).$$

Thus l_p does not have the (r, s) -DPP for $1 < r, s < \infty$.

Now, both (b) and (c) can be obtained on the lines of (a). \triangle

Note We have not been able to settle whether for $1 < p < \infty$, l_p has the 1-DPP.

4 Towards norm summability

In this section we shall examine a condition that forces every operator p -summable sequence to become norm- p -summable. Let X be a Banach space and $1 \leq p < \infty$. If $x \in l_p^w(X)$ is such that $E_x \in \Pi_p(l_{p'}, X)$, then it follows from Proposition 5.5(a) in [18] and by Proposition 2.4 that x is an operator p -summable sequence in X . In the light of this observation, we propose to study an operator version of the operator p -summable sequences.

Definition 4.1 *An operator $T \in \mathcal{B}(X, Y)$ is said to be p -limited if $T(Ball X)$ is p -limited in Y and T is said to be sequentially p -limited if $\langle Tx_n \rangle$ is operator p -summable for all $\langle x_n \rangle \in l_p^w(X)$.*

It follows from Proposition 2.4 that a sequence $\mathbf{x} = \langle x_n \rangle$ in X is operator p -summable if and only if $E_x \in \mathcal{B}(l_{p'}, X)$ is a p -limited operator if and only if $E_x \in \Pi_p^d(l_{p'}, X)$. Further, we have

Proposition 4.2 *Let $1 \leq p < \infty$. Every p -limited operator $T \in \mathcal{B}(X, Y)$ is sequentially p -limited.*

Proof. Let $\mathbf{x} = \langle x_n \rangle \in l_p^w(X)$. We may assume that $\|\langle x_n \rangle\|_p^w \leq 1$, so that $E_x(Ball(l_{p'})) \subset Ball(X)$. Since $T(Ball(X))$ is p -limited in Y . Thus $T(E_x(Ball(l_{p'})))$ is also p -limited in Y . Now by Lemma 2.2, $\langle Tx_n \rangle$ is operator p -summable in Y . \triangle

The following result will be used to characterize sequentially p -limited operator.

Lemma 4.3 *Let $1 \leq p < \infty$ and let $\alpha \in l_p^w(l_p)$. Then $\alpha \in l_p^s(l_p)$ if and only if $E_\alpha \in \Pi_p(l_{p'}, l_p) = \Pi_p^d(l_{p'}, l_p)$. Here $l_{p'} = c_0$ when $p = 1$.*

Proof. When $1 < p < \infty$, this fact follows from Remark (v) after Proposition 5.5 in [18]. Thus we may assume that $p = 1$. Again in this case, it follows, from [18, Proposition 5.5(a)], that if $E_\alpha \in \Pi_1(c_0, l_1)$, then $\alpha \in l_1^s(l_1)$.

Conversely, let $\alpha \in l_1^s(l_1)$, $\alpha = \langle \alpha_n \rangle = \langle \langle \alpha_n^k \rangle_k \rangle_n$ such that $\alpha_n = \langle \alpha_n^k \rangle_k \in l_1$ for all n . Put $\tilde{\alpha}_k = \langle \alpha_n^k \rangle_n$ for all k . Then

$$\sum_{k=1}^{\infty} \sum_{n=1}^{\infty} |\alpha_n^k| = \sum_{n=1}^{\infty} \sum_{k=1}^{\infty} |\alpha_n^k| = \|\alpha\|_1^s,$$

so that $\tilde{\alpha}_k \in l_1$ for all k with $\tilde{\alpha} = \langle \tilde{\alpha}_k \rangle \in l_1^s(l_1)$. If $\beta = \langle \beta_n \rangle = \langle \langle \beta_n^m \rangle_m \rangle_n \in l_1^w(c_0)$. Then

$$\begin{aligned} \langle E_\alpha(\beta_n) \rangle_n &= \left\langle \sum_{m=1}^{\infty} \beta_n^m \alpha_m \right\rangle_n \\ &= \left\langle \left\langle \sum_{m=1}^{\infty} \beta_n^m \alpha_m^k \right\rangle_k \right\rangle_n \\ &= \langle \langle (\beta_n, \tilde{\alpha}_k)_k \rangle_k \rangle_n. \end{aligned}$$

Since

$$\sum_{k=1}^{\infty} \sum_{n=1}^{\infty} |(\beta_n, \tilde{\alpha}_k)| \leq \|\beta\|_1^w \sum_{k=1}^{\infty} \|\tilde{\alpha}_k\| = \|\beta\|_1^w \|\alpha\|_1^s,$$

we get $\langle E_\alpha(\beta_n) \rangle_n \in l_1^s(l_1)$ with $\|\langle E_\alpha(\beta_n) \rangle_n\|_1^s \leq \|\beta\|_1^w \|\alpha\|_1^s$. Thus $E_\alpha \in \Pi_1(c_0, l_1)$ with $\pi_1(E_\alpha) \leq \|\alpha\|_1^s$. Since $\|\alpha\|_1^s \leq \pi_1(E_\alpha)$, we conclude $\pi_1(E_\alpha) = \|\alpha\|_1^s$. \triangle

Theorem 4.4 *Let $T \in \mathcal{B}(X, Y)$. For $1 \leq p < \infty$, the following are equivalent:*

- (1) *T is sequentially p -limited.*
- (2) *$TU \in \Pi_p^d(l_{p'}, Y)$ for all $U \in \mathcal{B}(l_{p'}, X)$.*
- (3) *$ST \in \Pi_p(X, l_p)$ for all $S \in \mathcal{B}(Y, l_p)$.*

Proof. That (1) is equivalent to (2) follows from Lemma 2.2.

Now let (1) hold. If $S \in B(Y, l_p)$ and if $x = \langle x_n \rangle \in l_p^w(X)$ so that $E_x \in B(l_{p'})$, then $TE_x \in \Pi_p^d(l_{p'}, X)$. Thus by Lemma 2.5, it follows that $STE_x \in \Pi_p^d(l_{p'}, l_p) = \Pi_p(l_{p'}, l_p)$. In other words, $\langle STx_n \rangle \in l_p^s(l_p)$ so that $ST \in \Pi_p(X, l_p)$. Thus (3) also holds. Finally, we can trace back the proof to show that (3) implies (1). \triangle

Remarks

- 1. If an operator $T \in \Pi_p(X, Y) \cup \Pi_p^d(X, Y)$, then T is sequentially p -limited.
- 2. Every sequentially p -limited operator in $\mathcal{B}(X, l_p)$ is in $\Pi_p(X, l_p)$.

Now we prove the following sequential characterization of subspaces of $L_p(\mu)$ whose operator characterization was obtained by Kwapién [12].

Theorem 4.5 *Let $1 \leq p < \infty$. For a Banach space X the following are equivalent:*

- (1) *Every operator p -summable sequence in X is norm p -summable.*

(2) $\Pi_p^d(Y, X) \subset \Pi_p(Y, X)$, for every Banach space Y .

(3) $\Pi_p^d(l_{p'}, X) = \Pi_p(l_{p'}, X)$.

(4) (Kwapién) X is a subspace of $L_p(\mu)$ for some Borel measure μ .

Proof. The equivalence of (2) and (4) was proved by Kwapién [12].

Let (1) hold. Assume that $T \in \Pi_p^d(Y, X)$ for some Banach space Y . If $y = \langle y_n \rangle \in l_p^w(Y)$, then $E_y \in B(l_{p'}, Y)$. Thus $TE_y \in \Pi_p^d(l_{p'}, X)$. Now, by Lemma 4 and assumption (1), we get that $\langle Ty_n \rangle \in l_p^{op}(X) = l_p^s(X)$. It follows that $T \in \Pi_p(Y, X)$ so that (2) holds.

Since $\Pi_p(l_{p'}, X) \subset \Pi_p^d(l_{p'}, X)$ follows from [18], we may conclude that (2) implies (3).

Finally, assume that (3) holds. Let $x \in l_p^{op}(X)$. Then $E_x \in \Pi_p^d(l_{p'}, X) = \Pi_p(l_{p'}, X)$. Now that $x \in l_p^s(X)$ again follows from [18]. This completes the proof. \triangle

Recall that a Banach space X is said to have the Gelfand-Phillips property if every limited set in X is relatively compact. Recall further that every limited set in a Banach space is conditionally weakly compact [3]. We do not know about the ‘ p -version’ of this result, possibly due to the absence of a p -prototype of a Rosenthal’s l_1 -theorem. At the same time let us note that in a Banach space, in which any operator p -summable sequence is norm p -summable, a (relatively) weakly p -compact set is (relatively) p -compact if and only if it is p -limited. Thus the condition that every operator p -summable sequence in a Banach space is norm p -summable can be seen as a p -version of the Gelfand-Phillips property.

An operator ideal: Let $1 \leq p < \infty$. For a pair of Banach spaces X and Y , consider the set $Lt_p(X, Y)$ of all sequentially p -limited operators in $B(X, Y)$. For $T \in Lt_p(X, Y)$, we define

$$lt_p(T) := \sup\{\pi_p(ST) : S \in B(Y, l_p) \text{ and } \|S\| \leq 1\}.$$

Then it is a routine to prove the following result.

Proposition 4.6 *For $1 \leq p < \infty$, (Lt_p, lt_p) is a normed operator ideal.*

Note We have not been able to show whether in general $Lt_p(X, Y)$ is lt_p -complete. We, however, adopted the following approach.

Let X and Y be Banach spaces and $T \in B(X, Y)$. For any $1 \leq p < \infty$, we can define $\varphi_T : B(Y, l_p) \rightarrow B(X, l_p)$ given by $\varphi_T(S) = ST$, for all

$S \in B(Y, l_p)$. Now it is easy to show that $T \mapsto \varphi_T$ is a linear isometry from $B(X, Y)$ into $B(B(Y, l_p), B(X, l_p))$; $1 \leq p \leq \infty$. Moreover, if $1 \leq p < \infty$, it follows from Proposition 4.3, that $T \in Lt_p(X, Y)$ if and only if $\varphi_T(B(Y, l_p)) \subset \Pi_p(X, l_p)$. In this case for all $T \in Lt_p(X, Y)$, we have

$$lt_p(T) = \text{The operator norm of } \varphi_T \text{ in } B(B(Y, l_p), \Pi_p(X, l_p)).$$

We write $Lt_p(X, Y)$ for the completion of $\{\varphi_T : T \in Lt_p(X, Y)\}$ in $B(B(Y, l_p), \Pi_p(X, l_p))$ and denote the operator norm on $Lt_p(X, Y)$ again by $lt_p(\cdot)$. Thus, proposition 4.11 may be re investigated in the following manner.

Proposition 4.7 *(Lt_p, lt_p) is a Banach operator ideal, $1 \leq p < \infty$.*

Remarks 1. If $x \in l_p^w(X)$ is operator p -summable, then $\|x\|_p^w = \|E_x\| \leq lt_p(E_x) := lt_p(x)$. If $x \in l_p^s(X)$, then $lt_p(x) \leq \|x\|_p^s$.
2. For $T \in \Pi_p(X, Y)$, $\|T\| \leq lt_p(T) \leq \pi(T)$. If $T \in \Pi_p(X, l_p)$, then $lt_p(T) = \pi_p(T)$.

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